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A STOCHASTIC, MULTISTAGE, MULTIPRODUCT INVESTMENT MODEL*

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Abstract. This paper presents a model for an n-stage multiproduct investment program. The problem of finding an "optimal" investment program is of great interest in industry.[1] Given a probabilistic estimate of future product(s) demand, we seek an optimum within a set of alternatives open to us. By the optimum we mean the minimum-minimorum of the total expected costs. The minimal cost and precise timing of the n stages are obtained by solving a set of functional equations using a combination of the recursive technique of dynamic programming and numerical methods.

1. Introduction. In this paper we discuss the formulation, analysis and solution of a mathematical model intended for use in selecting among alternative multistage investment programs. Such selection problems frequently confront both state planning agencies and private companies. The chosen program is to be that which is least costly (or most profitable) in the "present worth" sense.

The investments are to be used to provide capacity to meet future demands for one or more products of a given class. There may be significant penalties for either an excess or a deficit of capacity, the latter being met by "imports." Demands are regarded as "known" only in the form of probability distributions, so that it is really the expected value of discounted cost which is to be minimized.

The alternative programs will in general differ in their individual product capacities, investment requirements and operating costs. Some of them may involve the production of only a subset of the class of products being considered. Each alternative, however, must be capable of being "staged" over time. That is, facilities and capacity can be built up gradually (the model assumes discrete steps). All the "stages" of an alternative are required to be compatible; i.e., nonoperational combinations of stages are excluded.

The sequential stages of each alternative may be thought of as the sequential expansion of a multiproduct facility. In comparing alternatives

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one should associate, to each, that precise timing of its stages (i.e., the times when the stages should become operational) which minimizes the mean discounted cost. Thus the "best" alternative is selected as a minimum-minimorum.

Our effort is therefore focussed on the problem of determining the optimal time-phasing, for some one alternative which is completely specified *except* for this phasing. A mathematical model for this problem is developed in §2. Section 3 describes a solution method using the recursive technique of dynamic programming.

The solution has been applied to the selection of the optimal phasing over a fifteen-year planning period for twenty-seven different process alternatives for producing flat steel products (four in number). This was done for a South American country in which the steel facilities are state owned. The minimum-minimorum was determined under a variety of alternative assumptions on parameters such as interest rate, import prices, varying growth rates and assumed errors in these.

As it turned out the processes showed significant phasing sensitivity depending upon parameter values (eight sets of parameter values were tested). But under the four parameter alternatives considered most likely to prevail the minimum-minimorum was stable. Because of the lengthiness of the descriptive empirical material and results we do not include them in this paper. They are intended to be published elsewhere.

2. Formulation of model. The "present" will as usual be denoted by $t = 0$. We use the notation

$$T = \text{planning horizon},$$

$$n = \text{number of stages}.$$

The variables of our minimization problem are denoted by

$$(1) \qquad 0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n \leqq T,$$

where t_i is the time at which the ith stage begins. Note that the possible equalities in (1) admit simultaneous initiation of several stages.

For compactness, we set $\mathbf{t}_n = (t_1, \cdots, t_n)$ and let $\Delta(n, T)$ denote that portion of $\mathbf{t}_n$-space defined by (1). The minimization problem consists of determining

$$(2) \qquad f(n, T) = \min \{F^{(n)}(\mathbf{t}_n ; T) \mid \mathbf{t}_n \in \Delta(n, T)\},$$

and finding $\mathbf{t}_n^*$ at which the minimum is attained. We proceed to describe the explicit form of the function $F^{(n)}$.

The alternative whose timing is under study involves the quantities

$$K_i = \text{amount of capital invested at the beginning of the } i\text{th stage,}$$

i.e., at t_i.

Note that some of the K_i may be zero. With the notations

$$\alpha = \text{discounting rate,}$$

$$\beta = \text{salvage rate,}$$

we see that at time T the worth of the facilities introduced for the ith stage is $K_i \exp \{-\beta(T - t_i)\}$, which discounted back to t_i becomes

$$K_i \exp \{-(\beta + \alpha)(T - t_i)\}.$$

Hence effective net capital expended at t_i is

$$K_i - K_i \exp \{-(\beta + \alpha)(T - t_i)\},$$

whose contribution to the present worth function $F^{(n)}(\mathbf{t}_n ; T)$ is

$$K_i \exp (-\alpha t_i) - K_i \exp \{-(\beta + \alpha)T + \beta t_i\}.$$

Thus one summand of $F^{(n)}(\mathbf{t}_n ; T)$ is

$$\sum_{i=1}^{n} K_i[\exp (-\alpha t_i) - \exp \{-(\beta + \alpha)T + \beta t_i\}]. \tag{3}$$

The different products will be indexed $j = 1, \cdots, p$, where p denotes the total number of products. The investment policy alternative involves certain quantities

$$\rho_{ij} = \text{capacity for } j\text{th product in } i\text{th stage.}$$

The existence of the facilities involves certain fixed costs

$$d_{ij} = \text{unit (of capacity) cost associated with } j\text{th product in } i\text{th stage,}$$

while the operation of the facilities brings in variable costs

$$c_{ij} = \text{unit (of output) cost for } j\text{th product in } i\text{th stage.}$$

The demand functions

$$d_j(t) = \text{demand rate for } j\text{th product at time } t$$

are assumed to be exogeneous. Actual output rate is given by

$$r_{ij}(t) = \min (\rho_{ij}, d_j(t)), \qquad t \in [t_i, t_{i+1}], \tag{4}$$

with $t_{n+1} = T$. In other words, there is no "stockpiling." Note that some ρ_{ij} may be zero; in fact, we might for some j have *all* $\rho_{ij} = 0$.

The aforementioned costs lead to another summand of $F^{(n)}(\mathbf{t}_n ; T)$, namely,

$$E\left[\sum_{i=1}^{n}\sum_{j=1}^{p}\int_{t_i}^{t_{i+1}} \{c_{ij}\, r_{ij}(t) + d_{ij}\, \rho_{ij}\} \exp(-\alpha t)\, dt\right], \tag{5}$$

where the expectation operator E is required because the integrands r_{ij} involve the stochastic demand functions d_j.

The terms $d_{ij}\rho_{ij}$ in (5) impose a penalty for *excess* capacity. The remaining summand of $F^{(n)}$ expresses the cost of the "imports" (for a government actual imports, and for a private firm items purchased but not necessarily from abroad) required to compensate for *inadequate* capacity. Let

$$M_j = \text{delivered "import" unit price of } j\text{th product.}$$

The total discounted expenditure for imports is then given by

$$\sum_{j=1}^{p} M_j \int_0^{t_1} d_j(t) \exp(-\alpha t)\, dt + \sum_{i=1}^{n}\sum_{j=1}^{p} M_j \int_{t_i}^{t_{i+1}} [d_j(t) - r_{ij}(t)] \exp(-\alpha t)\, dt.$$

For underdeveloped countries, however, imports may be viewed as having "extra costs," dictated by the amount of foreign exchange available, national aspirations, etc. These "social costs" of imports are generally viewed as rising nonlinearly with the size of the import. We incorporate this by introducing a square term. The coefficients μ and λ of the penalty function (as well as the import prices M_j) could be treated as time-varying without any conceptual difficulties, but this would require unattainably complete knowledge of the future economy in the large, and much more cumbersome computations. Thus the final term of $F^{(n)}(\mathbf{t}_n ; T)$ reads

$$E\left\{\sum_{j=1}^{p} M_j \left[\int_0^{t_1} \{\mu\, d_j(t) + \lambda\, d_j^2(t)\} \exp(-\alpha t)\, dt + \sum_{i=1}^{n}\int_{t_i}^{t_{i+1}} \{\mu(d_j(t) - r_{ij}(t)) + \lambda(d_j(t) - r_{ij}(t))^2\} \exp(-\alpha t)\, dt\right]\right\}. \tag{6}$$

To summarize, the minimization problem in (2) takes the explicit form

$$f(n, T) = \min\{F^{(n)}(\mathbf{t}_n ; T) \mid \mathbf{t}_n \in \Delta(n, T)\}$$

$$
(7)\quad \begin{aligned}
&= \min_{\mathbf{t}_n \in \Delta(n,T)} \left\{ \sum_{i=1}^{n} K_i[\exp(-\alpha t_i) - \exp\{-(\beta+\alpha)T + \beta t_i\}] \right. \\
&+ E\left(\sum_{j=1}^{p} \left[M_j \int_0^{t_1} \{\mu d_j(t) + \lambda d_j^2(t)\} \exp(-\alpha t)\, dt \right.\right. \\
&+ \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \{c_{ij}\, r_{ij}(t) + d_{ij}\, \rho_{ij} \\
&\left.\left.\left. + M_j[\mu(d_j(t) - r_{ij}(t)) + \lambda(d_j(t) - r_{ij}(t))^2]\} \exp(-\alpha t)\, dt \right] \right) \right\}.
\end{aligned}
$$

3. Analysis. One now recognizes the possibility of applying some recursive technique such as that in Bellman [1]. With this purpose in mind, and regarding T as fixed, we define $G^{(n)}$ to be that part of $F^{(n)}$ arising from the interval $[0, t_n)$. Thus, for $0 \leqq t_n \leqq T$ and $\mathbf{t}_{n-1} \in \Delta(n-1, t_n)$, let

$$
(8)\quad \begin{aligned}
G^{(n)}(\mathbf{t}_{n-1}; t_n) &= \sum_{i=1}^{n-1} K_i\{\exp(-\alpha t_i) - \exp(-(\beta+\alpha)T + \beta t_i)\} \\
&+ E\left(\sum_{j=1}^{p} \left[\sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \{c_{ij}\, r_{ij}(t) + d_{ij}\, \rho_{ij} \right.\right. \\
&+ M_j[\mu(d_j(t) - r_{ij}(t)) + \lambda(d_j(t) - r_{ij}(t))^2]\} \exp(-\alpha t)\, dt \\
&\left.\left. + M_j \int_0^{t_1} \{\mu\, d_j(t) + \lambda\, d_j^2(t)\} \exp(-\alpha t)\, dt \right] \right),
\end{aligned}
$$

and set

$$
(9)\quad g(n, \tau) = \min\{G^{(n)}(\mathbf{t}_{n-1}; \tau) \mid \mathbf{t}_{n-1} \in \Delta(n-1, \tau)\}.
$$

It follows that

$$
(10)\quad \begin{aligned}
f(n, T) &= \min_{0 \leqq t \leqq T} \left\{ g(n, t) + K_n[\exp(-\alpha t) - \exp(-(\beta+\alpha)T + \beta t)] \right. \\
&+ E\left[\sum_{j=1}^{p} \int_t^T \{c_{nj}\, r_{nj}(s) + d_{nj}\, \rho_{nj} \right. \\
&+ M_j[\mu(d_j(s) - r_{nj}(s)) + \lambda(d_j(s) - r_{nj}(s))^2]\} \\
&\left.\left. \cdot \exp(-\alpha s)\, ds \right] \right\}.
\end{aligned}
$$

Thus the evaluation of $f(n, T)$ is reduced to a one-dimensional minimization problem—*if* we have an efficient method for calculating the values of $g(n, \tau)$. Such a method, however, can be arrived at by observing that the function $G^{(n)}$ obeys the recursion

$$
\begin{aligned}
G^{(n)}(\mathbf{t}_{n-1}\,;t_n) &= G^{(n-1)}(\mathbf{t}_{n-2}\,;t_{n-1}) \\
&+ K_{n-1}\{\exp(-\alpha t_{n-1}) - \exp(-(\beta+\alpha)T + \beta t_{n-1})\} \\
&+ E\Bigg[\sum_{j=1}^{p}\int_{t_{n-1}}^{t_n}\{c_{n-1,j}r_{n-1,j}(t) + d_{n-1,j}\rho_{n-1,j} \\
&+ M_j[\mu(d_j(t) - r_{n-1,j}(t)) \\
&\qquad + \lambda(d_j(t) - r_{n-1,j}(t))^2]\}\exp(-\alpha t)dt\Bigg]
\end{aligned}
\tag{11}
$$

This in turn gives us the functional equation

$$
\begin{aligned}
g(n,\tau) &= \min_{0\leqq t\leqq\tau}\Bigg\{g(n-1,t) \\
&+ K_{n-1}[\exp(-\alpha t) - \exp(-(\beta+\alpha)T + \beta t)] \\
&+ E\Bigg[\sum_{j=1}^{p}\int_{t}^{\tau}\{c_{n-1,j}r_{n-1,j}(s) + d_{n-1,j}\rho_{n-1,j} \\
&+ M_j[\mu(d_j(s) - r_{n-1,j}(s)) \\
&\qquad + \lambda(d_j(s) - r_{n-1,j}(s))^2]\}\exp(-\alpha s)\,ds\Bigg]\Bigg\}
\end{aligned}
\tag{12}
$$

for $n > 1$. The "boundary condition" corresponding to $n = 1$ is

$$
g(1,\tau) = E\left[\sum_{j=1}^{p} M_j\int_0^{\tau}\{\mu d_j(t) + \lambda d_j^2(t)\}\exp(-\alpha t)\,dt\right]. \tag{13}
$$

Thus the calculation of $f(n, T)$ can be carried out as a sequence of one-dimensional minimizations.

Further progress requires specifying the functional forms of the temporal and stochastic variations of the demands d_j. For initial simplicity, we assume positive constant growth rates b_j, which are themselves random variables. The more realistic nonconstant case is considered subsequently. Thus

$$
d_j(t) = d_j(t, b_j) = a_j\exp(b_j t), \qquad a_j, b_j > 0. \tag{14}
$$

This analytic form allows us to define $t_{ij}(b_j)$ uniquely by the specification

$$
d_j(t_{ij}(b_j), b_j) = \rho_{ij},
$$

which implies that

$$
t_{ij}(b_j) = b_j^{-1}\log(\rho_{ij}/a_j). \tag{15}
$$

We may abbreviate

$$t_{ij}(b) = t_{ij}(b_j), \tag{16}$$

since it is the function t_{ij} that we are defining. It follows from (4) that

$$d_j(s) - r_{ij}(s) = 0, \quad s \in [t_i, \min\{t_{ij}(b), t_{i+1}\}], \tag{17}$$

$$r_{ij}(s) = \rho_{ij}, \qquad s \in [\max\{t_i, t_{ij}(b)\}, t_{i+1}]. \tag{18}$$

With the aid of these, one verifies that the integral occurring in (12) can be rewritten as

$$\begin{aligned} \int_t^\tau \{\cdots\} \exp(-\alpha s)\, ds &= c_{ij} \int_t^{t'} c_j(s) \exp(-\alpha s)\, ds \\ &\quad + d_{ij}\rho_{ij} \int_t^\tau \exp(-\alpha s)\, ds + \rho_{ij}(c_{ij} + M_j(\lambda\rho_{ij} - \mu)) \\ &\quad \cdot \int_{t'}^\tau \exp(-\alpha s)\, ds + M_j(\mu - 2\lambda\rho_{ij}) \int_{t'}^\tau d_j(s) \exp(-\alpha s)\, ds \\ &\quad + \lambda M_j \int_{t'}^\tau d_j^2(s) \exp(-\alpha s)\, ds, \end{aligned} \tag{19}$$

where

$$t' = t'_{ij}(b) = \max\{t, \min(t_{ij}(b), \tau)\}. \tag{20}$$

With the aid of (14) the integrals in (19) can of course be evaluated explicitly.

It is convenient to write[2]

$$\begin{aligned} \Lambda(x, y, \theta) &= \int_x^y \exp\{(\theta - \alpha)s\}\, ds \\ &= (\alpha - \theta)^{-1} \{\exp[(\theta - \alpha)x] - \exp[(\theta - \alpha)y]\}. \end{aligned} \tag{21}$$

Then the right-hand side of (19) becomes

$$\begin{aligned} &c_{ij}a_j\Lambda(t, t', b) + d_{ij}\rho_{ij}\alpha^{-1}[\exp(-\alpha t) - \exp(-\alpha\tau)] \\ &\quad + \alpha^{-1}\rho_{ij}[c_{ij} + M_j(\lambda\rho_{ij} - \mu)][\exp(-\alpha t') - \exp(-\alpha\tau)] \\ &\quad + M_j(\mu - 2\lambda\rho_{ij})a_j\Lambda(t', \tau, b) + \lambda M_j a_j^2\Lambda(t', \tau, 2b). \end{aligned} \tag{22}$$

The expectation operator E (cf. (12)) must be taken into account. No explicit assumption about the probability distributions of $(b_1, \cdots, b_p)$ has yet been made. For computational facility we shall assume that the distributions $P_j(b)$ of the random variables b_j are statistically independent.

[2] The proper indeterminate form is assumed for $\alpha = \theta$.

Hence (12) takes the explicit form

$$
\begin{aligned}
g(i+1,\tau) = \min_{0\leq t\leq\tau} \Big\{ & g(i,t) + K_i\,[\exp(-\alpha t) \\
& - \exp(-(\beta+\alpha)T + \beta t)] + \sum_{j=1}^{p}\Big[c_{ij}a_j \int_{-\infty}^{\infty} \Lambda(t,t',b)P_j(b)\,db \\
& + d_{ij}\rho_{ij}\alpha^{-1}\{\exp(-\alpha t) - \exp(-\alpha\tau)\} + \alpha^{-1}\rho_{ij}\{c_{ij} \\
& + M_j(\lambda\rho_{ij} - \mu)\} \int_{-\infty}^{\infty} \{\exp(-\alpha t') - \exp(-\alpha\tau)\}P_j(b)\,db \\
& + M_j(\mu - 2\lambda\rho_{ij})a_j \int_{-\infty}^{\infty} \Lambda(t',\tau,b)P_j(b)\,db \\
& + \lambda M_j a_j^2 \int_{-\infty}^{\infty} \Lambda(t',\tau,2b)P_j(b)\,db \Big]\Big\} \\
= \min_{0\leq t\leq\tau} \varphi(i,t,\tau), &
\end{aligned} \tag{23}
$$

say, where t' was defined in (20).

To solve the phasing problem it is necessary to keep track of the points where the minimum of $\varphi(i, t, \tau)$ occurs. Let t^* be such that

$$g(i+1,\tau) = \varphi(i,t^*,\tau), \tag{24}$$

$$t^* = t^*(i,\tau). \tag{25}$$

Then in the n-stage problem, once T and n are given, $t^*(n, T)$ is the timing of the nth stage and inductively the timing of the $(n - 1)$th stage is $t^*(n - 1, t^*(n, T))$, and so on. Note that, though we explicitly defined $t^*(i, \tau)$ only for $i < n$, the obvious interpretation of $t^*(n, T)$ is used in connection with (10).

All real problems for which the preceding analysis would be useful are characterized by many investment alternatives and frequently by non-constant-growth-rates. Numerical solutions to such problems are feasible only through a digital computer, such as the CDC 3600 at Michigan State University, which was available to us. Though certain simplifications will result by assuming some reasonably simple forms for $P_j(b)$, still, the computations are much too long for a desk calculator—let alone the nonconstant-growth-rate-case.

We make the traditional but not unrealistic assumption that P_j's are normal distributions.[3] Explicitly,

$$P_j(b) = \frac{1}{\sigma_j\sqrt{2\pi}} \exp\left[-\frac{1}{2}(b-\bar{b}_j)^2/\sigma_j^2\right], \qquad j = 1, \cdots, p. \tag{26}$$

[3] In real situations the problem of the estimation of the parameters $\bar{b}_j$ and σ_j is

To evaluate the infinite integrals in (23) we make use of the well-known Gauss-Hermite quadrature formula (see [2]) which states that

$$(27)\qquad \int_{-\infty}^{\infty} \exp(-x^2) g(x)\, dx = \sum_{k=-m}^{m} H_k g(x_k) + \epsilon_m ,$$

where H_k are the Gauss-Hermite weights, x_k the corresponding abscissas and ϵ_m the error term. Using (26),

$$(28)\qquad \begin{aligned} \int_{-\infty}^{\infty} P_j(b) g(b)\, db &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-x^2) g(\bar{b}_j + \sqrt{2}\sigma_j x)\, dx \\ &\approx \sum_{k=-m}^{m} w_k g(s_{jk}), \end{aligned}$$

where

$$(29)\qquad s_{jk} = \bar{b}_j + \sqrt{2}\sigma_j x_k$$

and

$$(30)\qquad w_k = H_k/\sqrt{\pi}.$$

An appraisal of ϵ_m can be made by using a formula [2, p. 129], namely,

$$\epsilon_m = \frac{m!\sqrt{\pi}}{2^m(2m)!} g^{(2m)}(\eta).$$

In the present case a better estimate could be obtained by applying this formula to $\int_{-\infty}^{\infty} g_N(x) \exp(-x^2)\, dx$, where g_N is the truncation of g in the interval $[-N, N]$, and then making another error estimate for this truncation error using properties of $\exp(-x^2)$. Since the formula (27) is

of considerable importance. When there is lack of significant data to produce reasonable econometric forecasts, the possibility of estimating an optimistic, pessimistic and realistic rate of growth and then using a beta-distribution in place of the normal is not excluded. This procedure has been applied successfully in analogous situations in PERT. The attendant mathematical ramifications are evident. We have to replace the Gauss-Hermite weights and abscissas by appropriate Jacobi weights and abscissas. Of course, now the weights and abscissas themselves have to be computed by a computer routine each time since the associated Jacobi polynomials, and hence their roots, depend in a more complex way on the parameters of the beta-distribution than the Hermite polynomials on the parameters of the Gaussian distribution. A general computer program of this type would have the virtue of including such distributions as triangular and others also.

We shall present the normal case since all the remaining calculations but for the above noted initial difference are identical. Moreover, this difficulty just alluded to is somewhat routine.

exact for polynomials of degree $\leqq 2m - 1$ and since the function g is a combination of various exponentials we did not go through this procedure.

Some remarks are also in order regarding the possibility of s_{jk} becoming very negative which would be the case if we use large m. This occurs for large values of m, but in view of (17) and (18), the integrals in (19) would still be physically meaningful—provided the stochastic variations were estimated properly. The reason for this is that large negative values would occur with a low probability which would be taken into account by the quadrature formula. In our study referred to earlier, the values of σ_j and the chosen value of m were such that $s_{jk} \geqq 0$, for all j and k.

Thus, finally, (23) becomes

$$\begin{aligned} g(i+1, \tau) = \min_{0 \leqq t \leqq \tau} \{ & g(i, t) \\ & + K_i[\exp(-\alpha t) - \exp(-(\beta + \alpha)T' + \beta t)] \\ & + \alpha^{-1} \exp(-\alpha t) \sum_{j=1}^{p} d_{ij}\rho_{ij} - \alpha^{-1} \exp(-\alpha \tau) \sum_{j=1}^{p} \rho_{ij}\{d_{ij} \\ & + c_{ij} + M_j(\lambda\rho_{ij} - \mu)\} + \sum_{j=1}^{p} \sum_{k=-m}^{m} w_k[c_{ij}a_j\Lambda(t, t', s_{jk}) \\ & + \alpha^{-1}\rho_{ij}\{c_{ij} + M_j(\lambda\rho_{ij} - \mu)\} \exp(-\alpha t') \\ & + M_j a_j(\mu - 2\lambda\rho_{ij})\Lambda(t', \tau, s_{jk}) \\ & + \lambda M_j a_j^2 \Lambda(t', \tau, 2s_{jk})]\}, \end{aligned} \tag{31}$$

where

$$t' = t'_{ij}(s_{jk}), \tag{32}$$

and the formula (20) is used in its computation.

We may now discretize the problem in (31) in an obvious way by choosing a fine enough grid for the planning horizon $[0, T]$. The number of subdivisions of $[0, T]$ is mostly influenced by the physical nature of the problem. That is, a very fine subdivision resulting in possible additions of facilities at quick intervals may not be realizable. In fact, the subdivisions had to be spaced six months apart in our study. Thus it seems appropriate to view the grid-size as a constraint. Equations (31), (13), (10) and other related equations are now in a form suitable for translation into an algebraic language for use on a large-scale digital computer.

In the nonconstant-growth-rate case, we assume that the demand rates for each of the p products can be approximated in the intervals $[0, T_1]$,

$[T_1, T_2]$ and $[T_2, T]$ by

$$(33)\quad d_j(t, b_j) = \begin{cases} a_j \exp(b_j t), & 0 \leqq t \leqq T_1, \\ a_j \exp[b_j\{T_1 + \theta_j(t - T_1)\}], & T_1 \leqq t \leqq T_2, \\ a_j \exp[b_j\{T_1 + \theta_j(T_2 - T_1) + \bar{\theta}_j(t - T_2)\}], & \\ & T_2 \leqq t \leqq T. \end{cases}$$

Here the demand (a_j) for each product at $t = 0$ is assumed to be known with certainty, while b_j are stochastic variables. The interpretations of (10), (12) and (13) remain unaltered, but it is necessary to replace the simple formula (15). With this in view we define

$$(34)\quad \begin{aligned} t_{ij}^{(1)} &= t_{ij}^{(1)}(b) = b^{-1} \log(\rho_{ij}/a_j), \\ t_{ij}^{(2)} &= (\theta_j b)^{-1} \log(\rho_{ij}/a_j) - (1 - \theta_j)T_1/\theta_j \\ &= [t_{ij}^{(1)} - (1 - \theta_j)T_1]/\theta_j, \\ t_{ij}^{(3)} &= (\bar{\theta}_j b)^{-1} \log(\rho_{ij}/a_j) - (1 - \theta_j)T_1/\bar{\theta}_j - (\theta_j - \bar{\theta}_j)T_2/\bar{\theta}_j \\ &= \theta_j t_{ij}^{(2)}/\bar{\theta}_j - (\theta_j/\bar{\theta}_j - 1)T_2. \end{aligned}$$

The appropriate replacement of (15) defining $t_{ij}'(b)$ is then given by the following rule:

$$(35)\quad \begin{aligned} &\text{if } t_{ij}^{(1)} \leqq T_1, \quad \text{then} \quad t_{ij}(b) = t_{ij}^{(1)}; \\ &\text{if } T_1 < t_{ij}^{(1)}, \quad t_{ij}^{(2)} \leqq T_2, \quad \text{then} \quad t_{ij}(b) = t_{ij}^{(2)}; \\ &\text{if } T_1 < t_{ij}^{(1)} \quad \text{and} \quad T_2 < t_{ij}^{(2)}, \quad t_{ij}^{(3)} \leqq T, \quad \text{then} \quad t_{ij}(b) = t_{ij}^{(3)}; \\ &\text{if } T_1 < t_{ij}^{(1)}, \quad T_2 < t_{ij}^{(2)} \quad \text{and} \quad T < t_{ij}^{(3)}, \quad \text{then} \quad t_{ij}(b) = T. \end{aligned}$$

Equations (17), (18), (19) and (20) still apply. But (19) does not simplify to (22). To get analogous expressions we put

$$\tilde{a} = a_j \exp\{b(1 - \theta_j)T_1\}, \qquad \tilde{b} = b\theta_j,$$

and

$$(36)\quad \hat{a}_j = \tilde{a}_j \exp\{(\theta_j - \bar{\theta}_j)T_2\}, \qquad \hat{b} = b\bar{\theta}_j.$$

Let t' be defined via (35) and (20). Then the factor of c_{ij} in the first summand of (22) will be replaced by:

$$\begin{aligned} &a_j\Lambda(t, t', b) && \text{if } 0 \leqq t \leqq t' \leqq T_1, \\ &[a_j\Lambda(t, T_1, b) + \tilde{a}_j\Lambda(T_1, t', \tilde{b})] && \text{if } 0 \leqq t \leqq T_1 \leqq t' \leqq T_2, \\ &\tilde{a}_j\Lambda(t, t', \tilde{b}) && \text{if } T_1 \leqq t \leqq t' \leqq T_2, \end{aligned}$$

$$[a_j\Lambda(t, T_1, b) + \tilde{a}_j\Lambda(T_1, T_2, b) + \hat{a}_j\Lambda(T_2, t', b)] \quad \text{if } 0 \leqq t \leqq T_1 < T_2 \leqq t',$$

$$[\tilde{a}_j\Lambda(t, T_2, \tilde{b}) + \hat{a}_j\Lambda(T_2, t', \hat{b})] \quad \text{if } T_1 \leqq t \leqq T_2 \leqq t',$$

$$\hat{a}_j\Lambda(t, t', \hat{b}) \quad \text{if } T_2 \leqq t \leqq t'.$$

The two middle terms in (22) remain unaltered, but the last two terms must be replaced in a fashion similar to the above. In fact, if we change t to t' and t' to τ wherever they appear in this paragraph we get the appropriate multiplier of $M_j(\mu - 2\lambda\rho_{ij})$. To obtain the multiplier of λM_j (the last summand in (22)), we need to replace b by $2b$, a_j by a_j^2, $\tilde{b}$ by $2\tilde{b}$, $\tilde{a}_j$ by $\tilde{a}_j^2$, $\hat{b}$ by $2\hat{b}$ and $\hat{a}_j$ by $\hat{a}_j^2$ in the multiplier we have just obtained for the next to the last term of (22)—of course, we do not alter the definition of t'. With these modifications, but using the same $P_j(b)$'s, we arrive at a modified version of (31), the intermediate steps and reasoning being the same as before.

This is now ready for translation into a computer program. Though our actual computer program prints much further information which is traditionally sought in accounting and economic comparisons (consistent with our definition of optimality), we have suppressed these details since they are neither mathematically interesting nor directly related to our definition of optimality. The program written in Fortran 3600 for the nonconstant-growth-rate case will be published in another paper.

The authors are considering generalizations of the present model where stochastic demands of intermediate products are permitted. The solution of this problem, we believe, would be of considerable value in the optimal investment programming of whole sets of industries related in an input-output fashion.

REFERENCES

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